Current improvement for $B \to D^{(*)}\ell\bar{\nu}$ semileptonic decay using Oktay-Kronfeld action

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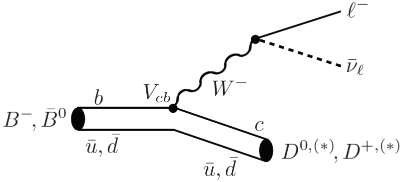
Exclusive determination of $|V_{cb}|$: $\bar{B} \to D^* \ell \bar{\nu}$ I



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$$\frac{d\Gamma}{dw}(\bar{B} \to D\ell\bar{\nu}) = \frac{G_F^2 m_D^3}{48\pi^3}(m_B + m_D)^2(w^2 - 1)^{3/2}\eta_{EW}^2|V_{cb}|^2\mathcal{G}(w)^2$$

$$\frac{d\Gamma}{dw}(\bar{B} \to D^*\ell\bar{\nu}) = \frac{G_F^2 m_{D^*}^3}{48\pi^3}(m_B - m_{D^*})^2(w^2 - 1)^{1/2}\eta_{EW}^2|V_{cb}|^2\chi(w)\mathcal{F}(w)^2$$

where the recoil parameter w is

$$w \equiv v_B \cdot v_{D^{(*)}} = \frac{m_B^2 + m_{D^{(*)}}^2 - q^2}{2m_B m_{D^{(*)}}}\left(= \frac{E_{D^{(*)}}}{m_{D^{(*)}}}, \quad \text{when } \bar{B} \text{ is at rest}\right)$$

Exclusive determination of $|V_{cb}|$: $\bar{B} \to D^* \ell \bar{\nu}$ II

- Lattice QCD : calculate hadronic matrix elements & determine $\mathcal{F}(w)$, $\mathcal{G}(w)$.

$$\frac{\langle D|V^\mu|\bar{B}\rangle}{\sqrt{m_B m_D}} = (v_B + v_D)^\mu h_+(w) + (v_B - v_D)^\mu h_-(w),$$

$$\frac{\langle D^*|V^\mu|\bar{B}\rangle}{\sqrt{m_B m_{D^*}}} = h_V(w)\epsilon^{\mu\nu\alpha\beta}\epsilon^{*\nu} v_{D^*}^\alpha v_B^\beta,$$

$$\frac{\langle D^*|A^\mu|\bar{B}\rangle}{\sqrt{m_B m_{D^*}}} = -i h_{A_1}(w)(w+1)\epsilon^{*\mu} + i h_{A_2}(w)(\epsilon^* \cdot v_B) v_B^\mu + i h_{A_3}(w)(\epsilon^* \cdot v_B) v_{D^*}^\mu.$$

At zero recoil ($v_B = v_{D^{(*)}}$, $w = 1$) $\mathcal{F}(1) = h_{A_1}(1)$.

- Determination of CKM matrix elements with lattice QCD

Decay rate (exp.) = known factors $\times |V_{CKM}|^2 \times$ Hadronic matrix elements

$\bar{B} \to D^* \ell \bar{\nu}$: error budget

- The error budget of calculation of the $\bar{B} \to D^* \ell \bar{\nu}$ semileptonic form factor at zero recoil by Fermilab/MILC [Fermilab/MILC (2014)]. The most dominant error is from the charm quark discretization.

form factor	h_{A_1}
decay channel	$\bar{B} \to D^* \ell \bar{\nu}$
statistics	0.4 %
matching	0.4 %
χPT	0.5 %
c discretization	1.0%
...	...
total	1.4%

$$h_{A_1}(1) \text{ discretization error} \approx \mathcal{O}(\alpha_s a \bar{\Lambda}^2 / m_c) + \mathcal{O}(\alpha_s a^2 \bar{\Lambda}^2) + \mathcal{O}(\bar{\Lambda}^3 / m_c^3)$$

with $\bar{\Lambda} \simeq m_B - m_b \approx \Lambda_{\text{QCD}}$.

How to control discretization error

- For physical system with heavy quarks (b and c), the discretization error can be controlled by using effective field theory framework. For the systems with heavy-light mesons, power counting parameters of the discretization error are λ and α_s.

$$\lambda \approx \Lambda_{\text{QCD}}/2m_Q \approx \Lambda_{\text{QCD}} a$$

- **Fermilab action** : improved up to $\mathcal{O}(\lambda)$.
  - Fermilab/MILC (2014) : $\bar{B} \to D^* \ell \bar{\nu}$ at zero recoil. Fermilab action and $\mathcal{O}(\lambda)$ improved currents for heavy quarks.
    - $\mathcal{O}(\lambda^2)$ mismatch in the action : has no effect at zero recoil.
    - $\mathcal{O}(\lambda^2)$ mismatch in the current : cancelled by taking ratio.

$$h_{A_1}(1) \text{ discretization error} \approx \mathcal{O}(\alpha_s \lambda^2) + \mathcal{O}(\lambda^3) \approx 1\%$$

How to control discretization error

- For physical system with heavy quarks (b and c), the discretization error can be controlled by using effective field theory framework. For the systems with heavy-light mesons, power counting parameters of the discretization error are λ and α_s.

$$\lambda \approx \Lambda_{\mathrm{QCD}}/2m_Q \approx \Lambda_{\mathrm{QCD}}a$$

- **Oktay-Kronfeld action(OK)** : improved up to $\mathcal{O}(\lambda^3)$.
  - SWME/LANL : OK action and $\mathcal{O}(\lambda^3)$ improved currents for heavy quarks.
  - Radiative correction
    - Action : improvement in $\mathcal{O}(\alpha\lambda)$ required, $\mathcal{O}(\alpha\lambda^2)$ recommended.
    - Current : improvement in $\mathcal{O}(\alpha\lambda)$, $\mathcal{O}(\alpha\lambda^2)$ required.

$$h_{A_1}(1) \text{ discretization error} \approx \mathcal{O}(\alpha_s\lambda^3) + \mathcal{O}(\lambda^4) \approx 0.2\%$$

Fermilab action

- The Fermilab action [El-Khadra, Kronfeld, and Mackenzie (1997)]

$$S_{\text{Fermilab}} \equiv S_0 + S_E + S_B,$$

$$S_0 \equiv a^4 \sum_x \bar{\psi}(x)\left[m_0 + \gamma_4 D_{\text{lat},4} - \frac{a}{2}\Delta_4 + \zeta\left(\boldsymbol{\gamma}\cdot\boldsymbol{D}_{\text{lat}} - \frac{r_s a}{2}\Delta^{(3)}\right)\right]\psi(x)$$

$$S_E \equiv -\frac{1}{2}c_E\zeta a^5 \sum_x \bar{\psi}(x)\boldsymbol{\alpha}\cdot\boldsymbol{E}_{\text{lat}}\psi(x), \quad S_B \equiv -\frac{1}{2}c_B\zeta a^5 \sum_x \bar{\psi}(x)i\boldsymbol{\Sigma}\cdot\boldsymbol{B}_{\text{lat}}\psi(x),$$

$\Delta^{(3)}, \Delta_4$: discretized versions of $\boldsymbol{D}^2$, D_4^2.

- Coefficients in the action $(m_0, c_B, \cdots)$ are tuned to reduce error in $\mathcal{O}(\lambda)$.

$$\mathcal{L}_{\text{Fermilab}} \doteq \bar{h}^+\left[-D_4 - m_1 + \frac{1}{2m_2}\boldsymbol{D}^2 + \frac{i}{2m_b}\boldsymbol{\sigma}\cdot\boldsymbol{B}\right]h^+ + \cdots,$$

  - m_2 (kinetic mass) : adjusted to the physical mass.
  - m_B : adjusted physically $\frac{1}{2m_B} \to \frac{z_B}{2m} = \frac{1+\mathcal{O}(\alpha)}{2m}$.
    (For tree-level matching, $c_B = r_s$ gives $m_B = m_2$)
  - m_1 : no effect on energy splitting, matrix elements.

Oktay-Kronfeld action I

- Oktay-Kronfeld (OK) action [Oktay and Kronfeld (2008)]

$$S_{\mathrm{OK}} \equiv S_0 + S_B + S_E + S_6 + S_7$$

$$S_6 \equiv a^6 \sum_x \bar{\psi}(x) \Big[c_1 \sum_i \gamma_i D_{\mathrm{lat},i} \Delta_i + c_2 \{ \boldsymbol{\gamma} \cdot \boldsymbol{D}_{\mathrm{lat}}, \Delta^{(3)} \} + c_3 \{ \boldsymbol{\gamma} \cdot \boldsymbol{D}_{\mathrm{lat}}, i \boldsymbol{\Sigma} \cdot \boldsymbol{B}_{\mathrm{lat}} \} + c_{EE} \{ \gamma_4 D_{\mathrm{lat},4}, \boldsymbol{\alpha} \cdot \boldsymbol{E}_{\mathrm{lat}} \} \Big] \psi(x),$$

$$S_7 \equiv a^7 \sum_x \bar{\psi}(x) \sum_i \Big[c_4 \Delta_i^2 \psi(x) + c_5 \sum_{j \neq i} \{ i \Sigma_i B_{\mathrm{lat},i}, \Delta_j \} \Big] \psi(x).$$

- c_i are fixed by tree-level matching
  - Dispersion relation : match quark propagator. (fix c_1, c_2, and c_4)
  - Interaction with background field : vertex with one gluon emission. (fix c_B, c_E, c_3 and c_5)
  - Compton scattering : vertex with two gluon emissions. (fix c_{EE})

Oktay-Kronfeld action II

- The lattice cutoff effect is factorized into short-distance coefficients which are symbolized as mass-like terms

$$\mathcal{L}_{OK} \doteq \bar{h}^{+}\Big[- D_4 - m_1 + \frac{1}{2m_2}\boldsymbol{D}^2 + \frac{i}{2m_B}\boldsymbol{\sigma}\cdot\boldsymbol{B} + \frac{\boldsymbol{D}\cdot\boldsymbol{E} - \boldsymbol{E}\cdot\boldsymbol{D}}{8m_E^2}$$
$$+ \frac{i\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{D})}{8m_E^2} + \frac{\boldsymbol{D}^4}{8m_4^3} + \frac{\{\boldsymbol{D}^2, i\boldsymbol{\sigma}\cdot\boldsymbol{B}\}}{8m_{B'}^3} - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{E})^2 + (\boldsymbol{\sigma}\cdot\boldsymbol{B})^2}{8m_{EE}^3}\Big] h^{+} + \cdots$$

- The matching is done analytically by adjusting all the mass-like terms to be identical to m_2. For example,

$$\frac{1}{4m_E^2 a^2} = \frac{\zeta^2}{\big[m_0 a(2 + m_0 a)\big]^2} + \frac{\zeta^2 c_E}{m_0 a(2 + m_0 a)},$$

and the condition that $m_E = m_2$ (physical mass) fixes c_E

$$c_E = \frac{\zeta^2 - 1}{m_0 a(2 + m_0 a)} + \frac{r_s \zeta}{1 + m_0 a} + \frac{r_s^2 m_0 a(2 + m_0 a)}{4(1 + m_0 a)^2}.$$

Matching with HQET framework

- The QCD flavor-changing axial-current $\mathcal{A}_\mu$ is matched to the HQET axial-current as

$$\mathcal{A}_\mu \doteq A_\mu^{(0)} + A_\mu^{(1)} + A_\mu^{(2)} + \cdots ,$$

where, $A_\mu^{(i)}$ for $i = 0, 1, 2, \cdots$ is i-th order HQET current.

For example, the leading order HQET current is given by

$$A_\mu^{(0)} = \eta_A \bar{h}^c_{v'} \gamma_\mu \gamma_5 h^b_v - \frac{1}{2} \beta_A (v' - v)_\mu \bar{h}^c_{v'} \gamma_5 h^b_v - \frac{1}{2} \gamma_A (v' - v)^\nu \bar{h}^c_{v'} i \sigma_{\mu\nu} \gamma_5 h^b_v$$

- The matrix element for $\bar{B} \to D^* \ell \bar{\nu}$ is given by

$$\langle D^* | \mathcal{A}_\mu | B \rangle = \langle D^*_{v'} | A_\mu^{(0)} | B_v \rangle + \langle D^*_{v'} | A_\mu^{(1)} | B_v \rangle + \langle D^*_{v'} | A_\mu^{(2)} | B_v \rangle + \cdots$$

where $v = p_B / m_B$ and $v' = p_{D^*} / m_{D^*}$.

Matching with HQET framework

- The meson states with full HQET theory can be expanded in terms of the meson states in the heavy quark limit [Kronfeld (2000)].

$$\frac{\langle D_{v'}|TO_1\cdots O_n|B_v\rangle}{\langle D_{v'}|D_{v'}\rangle^{1/2}\langle B_v|B_v\rangle} = \frac{\langle D^{(0)}_{v'}|TO_1\cdots O_n e^{\int d^4x\mathcal{L}_I}|B^{(0)}_v\rangle}{\langle D^{(0)}_{v'}|Te^{\int d^4x\mathcal{L}^c_I}|D^{(0)}_{v'}\rangle^{1/2}\langle B^{(0)}_v|Te^{\int d^4x\mathcal{L}^b_I}|B^{(0)}_v\rangle^{1/2}} \tag{1}$$

where $\mathcal{L}_I = \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \cdots$ is the higher order HQET Lagrangians

$$\mathcal{L}^{(1)} = \frac{1}{2m}\bar{h}_v\left(D^2_\perp + z_B s_{\mu\nu}B^{\mu\nu}\right)h_v$$

$$\mathcal{L}^{(2)} = \frac{1}{8m^2}\bar{h}_v\left(z_D[D^\alpha_\perp, iE_\alpha] + z_E s_{\alpha\beta}\{D^\alpha_\perp, iE^\beta\}\right)h_v$$

with $s_{\alpha\beta} = -i\sigma_{\alpha\beta}/2$.

- $|B_v\rangle$: states from theory with full Lagrangian $\mathcal{L}_{\text{Full}} = \mathcal{L}_{\text{light}} + \mathcal{L}^{(0)} + \mathcal{L}_I$
- $|B^{(0)}_v\rangle$: states from theory with the leading HQET Lagrangian $\mathcal{L}_{\text{H.L.}} = \mathcal{L}_{\text{light}} + \mathcal{L}^{(0)}$.

Matching with HQET framework

- For example, the contribution from leading order current can be expanded as

$$
\begin{aligned}
\langle D^*_{v'}|A^{(0)}_\mu|B_v\rangle &= \langle D^{*(0)}_{v'}|A^{(0)}_\mu|B^{(0)}_v\rangle \\
&+ \int d^4x \langle D^{*(0)}_{v'}|TA^{(0)}_\mu \mathcal{L}^{(1)}_b(x)|B^{(0)}_v\rangle^\star + \int d^4x \langle D^{*(0)}_{v'}|T\mathcal{L}^{(1)}_c(x)A^{(0)}_\mu|B^{(0)}_v\rangle^\star \\
&+ \int d^4x \langle D^{*(0)}_{v'}|TA^{(0)}_\mu \mathcal{L}^{(2)}_b(x)|B^{(0)}_v\rangle^\star + \frac{1}{2}\int d^4x\, d^4y \langle D^{*(0)}_{v'}|TA^{\mu(0)}\mathcal{L}^{(1)}_b(x)\mathcal{L}^{(1)}_b(y)|B^{(0)}_v\rangle^\star \\
&+ \int d^4x \langle D^{*(0)}_{v'}|T\mathcal{L}^{(2)}_c(x)A^{(0)}_\mu|B^{(0)}_v\rangle^\star + \frac{1}{2}\int d^4x\, d^4y \langle D^{*(0)}_{v'}|T\mathcal{L}^{(1)}_c(x)\mathcal{L}^{(1)}_c(y)A^{(0)}_\mu|B^{(0)}_v\rangle^\star \\
&+ \int d^4x\, d^4y \langle D^{*(0)}_{v'}|T\mathcal{L}^{(1)}_c(x)A^{(0)}_\mu\mathcal{L}^{(1)}_b(y)|B^{(0)}_v\rangle^\star + \mathcal{O}(\bar{\Lambda}^3/m_c^3).
\end{aligned}
$$

  where $(\bar{\Lambda} \approx m_D - m_c \approx m_B - m_b)$.
  ($\star$: reminder to include terms generated by expanding the denominator of Eq. (1))
- The heavy quark's spin and flavor symmetry gives strong constraints on the matrix elements on the R.H.S of the above equations.

Matching with HQET framework

- At zero recoil $v' = v$,

$$\langle D_v^* | A_\mu^{(0)} | B_v \rangle = \eta_A \epsilon_\mu^* \mathcal{W}_{01}^{(0)}$$

where ϵ_μ^* is the polarization of D^* and

$$\mathcal{W}_{01}^{(0)} = 1 - \frac{1}{2}\Delta_2(\Delta_2 D - 2\Theta_B E) - \frac{1}{2}\Delta_B(\Delta_B R_1 - \Theta_B R_2) - \frac{z_{Bc} z_{Bb}}{2m_c 2m_b}\left(\frac{4}{3}R_1 + 2R_2\right) + \cdots ,$$

the coefficients of the HQET action are factorized into

$$\Delta_2 = \frac{1}{2m_c} - \frac{1}{2m_b}, \quad \Delta_B = \frac{z_{Bc}}{2m_c} - \frac{z_{Bb}}{2m_b}, \quad \Theta_B = \frac{z_{Bc}}{2m_c} + \frac{3z_{Bb}}{2m_b},$$

D, E, R_1, R_2 is $\bar{\Lambda}^2$ order HQET matrix elements ($\bar{\Lambda} \approx m_B - m_b$).

Matching with HQET framework

- Similarly, the lattice counterpart is

$$\langle D_v^* | A_{\text{lat}}^{(0)} | B_v \rangle = \eta_A^{\text{lat}} W_{01}^{(0)}$$

where

$$W_{01}^{(0)} = 1 - \frac{1}{2}\Delta_2^{\text{lat}}(\Delta_2^{\text{lat}} D - 2\Theta_B^{\text{lat}} E) - \frac{1}{2}\Delta_B^{\text{lat}}(\Delta_B^{\text{lat}} R_1 - \Theta_B^{\text{lat}} R_2) + \cdots$$
$$- \frac{1}{2m_{Bc} 2m_{Bb}}\left(\frac{4}{3}R_1 + 2R_2\right)$$

$$\Delta_2^{\text{lat}} = \frac{1}{2m_{2c}} - \frac{1}{2m_{2b}}, \quad \Delta_B^{\text{lat}} = \frac{1}{2m_{Bc}} - \frac{1}{2m_{Bb}}, \quad \Theta_B^{\text{lat}} = \frac{1}{2m_{Bc}} + \frac{3}{2m_{Bb}}.$$

- Thus, the discretization error of the matrix element comes from the discrepancy in the short-distance coefficients Δ_2, Δ_B, and Θ_B.

$\bar{B} \to D^* \ell \bar{\nu}$ at zero recoil by FNAL/MILC

- The expansion of the matrix elements in the second order of $1/m_Q$

$$\langle D^*_{\text{lat}} | Z_{A_{cb}} \epsilon \cdot A_{\text{lat}} | B_{\text{lat}} \rangle = \langle D^*_v | \epsilon \cdot A^{(0)}_{\text{lat}} | B_v \rangle + \langle D^*_v | \epsilon \cdot A^{(2)}_{\text{lat}} | B_v \rangle$$
$$= Z_{A_{cb}} \eta^{\text{lat}}_{A_{cb}} W^{(0)}_{01} + W^{(2)}_{01},$$

the errors come from

- $\eta^{\text{lat}}_{A_{cb}}$: mismatch in the coefficient of leading order current. It is controlled by $Z_{A_{cb}}$.
- $W^{(0)}_{01}$: mismatch in the matched Lagrangian. At zero recoil, it has advantages that $\langle A^{(0)} \mathcal{L}^{(1)} \rangle = \langle A^{(0)} \mathcal{L}^{(2)} \rangle = 0$. Only $\langle A^{(0)} \mathcal{L}^{(1)} \mathcal{L}^{(1)} \rangle$ contributes.
- $W^{(1)}_{01} = 0$ at zero recoil.
- $W^{(2)}_{01}$: mismatch in the matching of second order HQET current $\to$ leading discrepancy is cancelled by taking ratio (Hashimoto ratio)

Simulation with the Oktay-Kronfeld action

- Using the Oktay-Kronfeld heavy quarks, we expect the discretization errors from hadronic states to be reduced as $\mathcal{O}(\lambda^4)$.
- Without $\mathcal{O}(\lambda^3)$-improved current, however, the discretization error from the mismatch in the lattice current is not under controlled.

$$\delta_{\text{dis.err}}\Big(\langle D^*_{\text{lat}}|Z_{A_{cb}}\epsilon\cdot A_{\text{lat}}|B_{\text{lat}}\rangle\Big)$$
$$= \delta\langle D^*_v|\epsilon\cdot A^{(0)}_{\text{lat}}|B_v\rangle + \delta\langle D^*_v|\epsilon\cdot A^{(1)}_{\text{lat}}|B_v\rangle + \delta\langle D^*_v|\epsilon\cdot A^{(2)}_{\text{lat}}|B_v\rangle + \delta\langle D^*_v|\epsilon\cdot A^{(3)}_{\text{lat}}|B_v\rangle$$

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$$\to \mathcal{O}(\alpha_s\lambda^2) + \mathcal{O}(\alpha_s\lambda^2) + \mathcal{O}(\lambda^2) + \mathcal{O}(\lambda^3) \qquad \text{use } \mathcal{O}(\lambda)\text{-improved current}$$

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$$\to \mathcal{O}(\alpha_s^2\lambda^2) + \mathcal{O}(\alpha_s^2\lambda^2) + \mathcal{O}(\alpha_s^2\lambda^2) + \mathcal{O}(\lambda^4) \qquad \text{one loop improve for } c_B,\ A^{(1)}_{\text{lat}}, A^{(2)}_{\text{lat}}$$

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$$\to \mathcal{O}(\alpha_s^2\lambda^2) + \mathcal{O}(\alpha_s^2\lambda^2) + \mathcal{O}(\alpha_s^2\lambda^2) + \mathcal{O}(\lambda^4) \qquad \text{one loop improve for } c_B,\ A^{(1)}_{\text{lat}}, A^{(2)}_{\text{lat}}$$

- $\mathcal{O}(\lambda^3)$ improvement of the lattice current is a first step for simulations with the OK action.

$\mathcal{O}(\lambda)$-improved current

- Improved currents are constructed by improved quark fields [El-Khadra, Kronfeld, and Mackenzie (1997)]

$$V_\mu^{\text{lat}} = \bar{\Psi}_c \gamma_\mu \Psi_b, \qquad A_\mu^{\text{lat}} = \bar{\Psi}_c \gamma_\mu \gamma_5 \Psi_b,$$

where

$$\Psi_f = e^{m_{1f}a/2}(1 + d_{1f}\, a\gamma \cdot \boldsymbol{D}_{\text{lat}})\psi_f, \quad f = b, c$$

with the tree-level rest mass $m_1 a = \log(1 + m_0 a)$.

- The current is matched to HQET current as (with $v = v' = (1, \mathbf{0})$)

$$V_\mu^{\text{lat}} \doteq \bar{h}_v^c \gamma_\mu h_v^b - \frac{1}{2m_{3b}} \bar{h}_v^c \gamma_\mu \boldsymbol{D} h_v^b + \frac{1}{2m_{3c}} \bar{h}_v^c \overleftarrow{\boldsymbol{D}} \gamma_\mu h_v^b + \cdots$$

$$A_\mu^{\text{lat}} \doteq \bar{h}_v^c \gamma_\mu \gamma_5 h_v^b - \frac{1}{2m_{3b}} \bar{h}_v^c \gamma_\mu \gamma_5 \boldsymbol{D} h_v^b + \frac{1}{2m_{3c}} \bar{h}_v^c \overleftarrow{\boldsymbol{D}} \gamma_\mu \gamma_5 h_v^b + \cdots$$

at the tree level

$$\frac{1}{2m_{3f}a} = \frac{\zeta(1 + m_{0f}a)}{m_{0f}a(2 + m_{0f}a)} - d_{1f}, \qquad (f = b, c)$$

the condition that $m_3 = m_2$ (kinetic quark mass) determines d_1.

$\mathcal{O}(\lambda^3)$ current improvement

- For the current improvement up to $\mathcal{O}(\lambda^3)$, we adopt the idea of improved quark field and extend it to the higher order.

$$\Psi_f = e^{m_{1f}a/2}(1 + d_{1f}a\gamma \cdot \boldsymbol{D}_{\text{lat}} + \cdots)\psi_f, \quad f = b, c$$

- Considering only tree level, the HQET Lagrangian can be derived by the Foldy-Wouthouysen-Tani (FWT) transformation which eliminates interaction between quark h^+ and anti-quark h^- in the Lagrangian

$$\mathcal{L}_{\text{QCD}} = -\bar{q}[\not{D} + m]q \rightarrow \mathcal{L}_{\text{HQ}},$$

by

$$q \rightarrow \left[1 - \frac{\gamma \cdot \boldsymbol{D}}{2m} + \cdots\right]h, \qquad h = h^+ + h^-,$$

the tree-level relation between QCD operator and the HQET operator is simply given by the FWT transformation

$$\bar{c}\gamma_\mu b \doteq \bar{h}_c\gamma_\mu h_b - \bar{h}_c\gamma_\mu\frac{\gamma \cdot \boldsymbol{D}}{2m_b}h_b + \bar{h}_c\frac{\gamma \cdot \overleftarrow{\boldsymbol{D}}}{2m_c}\gamma_\mu h_b + \cdots$$

$\mathcal{O}(\lambda^3)$ current improvement

- Taking FWT transformation through $\mathcal{O}(1/m_q^3)$ as ansatz, we introduced improved quark field

$$
\begin{aligned}
\Psi(x) =& e^{m_1 a/2}\Big[1 + d_1 a\boldsymbol{\gamma}\cdot\boldsymbol{D}_{\text{lat}} \quad \rightarrow \mathcal{O}(\lambda)\text{ improve} \\
&+ \frac{1}{2}d_2 a^2\Delta^{(3)} + \frac{1}{2}i d_B a^2\boldsymbol{\Sigma}\cdot\boldsymbol{B}_{\text{lat}} + \frac{1}{2}d_E a^2\boldsymbol{\alpha}\cdot\boldsymbol{E}_{\text{lat}} \quad \rightarrow \mathcal{O}(\lambda^2)\text{ improve} \\
&+ \frac{1}{6}d_3 a^3\gamma_i D_{\text{lat}i}\Delta_i + d_{EE} a^3\{\gamma_4 D_{4\text{lat}}, \boldsymbol{\alpha}\cdot\boldsymbol{E}_{\text{lat}}\} + d_{r_E} a^3\{\boldsymbol{\gamma}\cdot\boldsymbol{D}_{\text{lat}}, \boldsymbol{\alpha}\cdot\boldsymbol{E}_{\text{lat}}\} \\
&+ \frac{1}{2}d_4 a^3\{\boldsymbol{\gamma}\cdot\boldsymbol{D}_{\text{lat}}, \Delta^{(3)}\} + d_5 a^3\{\boldsymbol{\gamma}\cdot\boldsymbol{D}_{\text{lat}}, i\boldsymbol{\Sigma}\cdot\boldsymbol{B}_{\text{lat}}\} \quad \rightarrow \mathcal{O}(\lambda^3)\text{ improve} \\
&+ d_6 a^3[\gamma_4 D_{4\text{lat}}, \Delta^{(3)}] + d_7 a^3[\gamma_4 D_{4\text{lat}}, i\boldsymbol{\Sigma}\cdot\boldsymbol{B}_{\text{lat}}]\Big]\psi(x).
\end{aligned}
$$

- The term with d_3 is necessary to remedy rotational symmetry breaking of lattice quark.
- We should fix the improvement parameters d_i by matching.

Matching condition

- Matching two-quark matrix elements,

$$\langle c(\boldsymbol{p}_c, s_c)|\bar{\Psi}_c \Gamma \Psi_b|b(\boldsymbol{p}_b, s_b)\rangle_{\text{lat}} = \langle c(\boldsymbol{p}_c, s_c)|\bar{c}\Gamma b|b(\boldsymbol{p}_b, s_b)\rangle,$$

with $\Gamma = \gamma_\mu, \gamma_\mu \gamma_5$.

  - Straightforward extension of calculation in [El-Khadra, Kronfeld, and Mackenzie (1997)].
  - At tree-level, it gives constraint only on d_1, d_2, d_3, and d_4.

- Matching four-quark matrix elements,

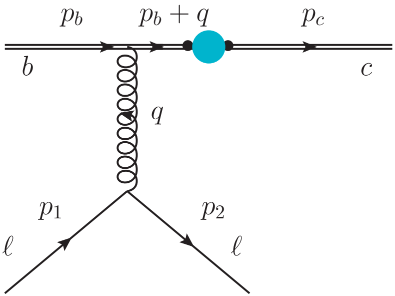
$$\langle \ell(\boldsymbol{p}_2, s_2)c(\boldsymbol{p}_c, s_c)|\bar{\Psi}_c \Gamma \Psi_b|b(\boldsymbol{p}_b, s_b)\ell(\boldsymbol{p}_1, s_1)\rangle_{\text{lat}}, \\ = \langle \ell(\boldsymbol{p}_2, s_2)c(\boldsymbol{p}_c, s_c)|\bar{c}\Gamma b|b(\boldsymbol{p}_b, s_b)\ell(\boldsymbol{p}_1, s_1)\rangle,$$

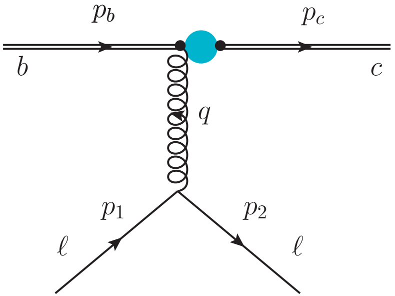
where ℓ represents a light spectator quark.

  - Process with one gluon exchange is necessary to obtain constraints on the improvement parameters other than d_{1-4}.
  - Only focus on the heavy quark part

Matching

- The lattice Feynman diagrams with gluon exchange at the *b*-quark line are as follows. The colored circle represents the flavor-exchanging operator.





- The colored circle represents the flavor-exchanging operator.
- The small dots with and without gluon line represent zero-gluon and one-gluon vertex from improved quark field, respectively.

Matching

- Let us consider matching condition from the sub-diagram with one-gluon exchange at the b-quark line,

$$n_\mu(q)\Big[R_b^{(0)}(p_b+q)S^{\text{lat}}(p_b+q)(-gt^a)\Lambda_{b\mu}(p_b+q,p_b) + (-gt^a)R_{b\mu}^{(1)}(p_b+q,p_b)\Big]\mathcal{N}_b(\boldsymbol{p}_b)u_b^{\text{lat}}(\boldsymbol{p}_b,s_b) = S(p_b+q)\gamma_\mu(-gt^a)\sqrt{\frac{m_b}{E_b}}u_b(\boldsymbol{p}_b,s_b). \tag{2}$$

  - $n_\mu(q)$: wave-function factor for lattice gluon.
  - $\Lambda_{b\mu}$: one-gluon vertex of OK-action.
  - $\mathcal{N}_b u_b^{\text{lat}}$: lattice spinor of the OK action.
  - $S^{\text{lat}}(p_b+q)$ and $S(p_b+q)$: quark propagator for lattice and continuum, respectively.
  - $R_b^{(0)}$, $R_{b\mu}^{(1)}$: zero-gluon and one-gluon vertex for improved field which contains improvement coefficients d_i.

Matching

- Expand the both side of the matching condition up to λ^3 order with the assumption

$$\boldsymbol{p}_b, \boldsymbol{q}, q_4 \sim \Lambda_{\rm QCD},$$

then $\boldsymbol{p}_b/m, q_\mu/m$ (in R.H.S) and $\boldsymbol{p}_b a, q_\mu a$ (in L.H.S.) are order of λ.

- Expansion of the quark propagator

$$\begin{aligned} S(p_b+q) &= \frac{m_b - i\gamma\cdot(p_b+q)}{m_b^2+(p_b+q)^2} \\ &= \frac{m_b(1+\gamma_4) - i\gamma_4(\tilde{p}_4+q_4) - i\boldsymbol{\gamma}\cdot(\boldsymbol{p}_b+\boldsymbol{q})}{2im_b(\tilde{p}_4+q_4)+(\tilde{p}_4+q_4)^2+(\boldsymbol{p_b}+\boldsymbol{q})^2} \\ &= \frac{(1+\gamma_4)}{2}\frac{1}{i(\tilde{p}_4+q_4)} + \cdots \end{aligned}$$

where

$$\tilde{p}_4 = p_{b,4} - im_b = i(E_b - m_b) = i\left[\frac{\boldsymbol{p}_b^2}{2m_b} - \frac{(\boldsymbol{p}_b^2)^2}{8m_b^2}\right] + \cdots,$$

the leading term is identical to the HQET propagator.

Matching

- The tree-level propagator of the OK quark can be expanded in similar way

$$S^{\text{lat}}(p_b+q) = \frac{1}{i\gamma_\mu K_\mu(p_b+q) + L(p_b+q)} = e^{-m_{1b}} \frac{(1+\gamma_4)}{2} \frac{1}{i(\tilde{p}_4^{\text{lat}} + q_4)} + \cdots$$

with

$$L(p) = 1 + m_0 a + \frac{1}{2} r_s \zeta \hat{\boldsymbol{p}}^2 a^2 + c_4 \sum_i (\hat{p}_i a)^4 - \cos(p_4 a)$$

$$K_i(p) = \sin(p_i a)[\zeta - 2c_2 \hat{\boldsymbol{p}}^2 a^2 - c_1 \hat{p}_i^2 a^2], \quad K_0(p) = \sin(p_4 a)$$

where $\hat{p}_i = 2\sin(p_i a/2)$.

$$\tilde{p}_4^{\text{lat}} = p_{b,4}^{\text{lat}} - i m_{1b} = i\left[\frac{\boldsymbol{p}_b^2}{2m_{2b}} - \frac{1}{6} w_{4b} a^3 \sum_i p_i^4 - \frac{(\boldsymbol{p}^2)^2}{8m_{4b}^3}\right] + \cdots,$$

$\tilde{p}_4 = \tilde{p}_4^{\text{lat}}$ by the action matching $(w_{4b} = 0, m_{2b} = m_{4b} = m_b)$.

- Expand both side of the sub-diagram through λ^3 is as follow (drop b-index)

$$\text{R.H.S } (\mu = 4) = \left[\frac{1}{iP_4} - \frac{\gamma \cdot (\boldsymbol{p} + \boldsymbol{q})}{2mP_4} + \frac{(\boldsymbol{p} + \boldsymbol{q})^2}{2mP_4^2} - \frac{i\gamma \cdot \boldsymbol{q}}{4m^2} + \epsilon_{ijk} \Sigma_i \frac{q_j p_k}{4m^2 P_4} \right.$$
$$\left. + \frac{i\left(\boldsymbol{p}^2 + 2\boldsymbol{q} \cdot (\boldsymbol{p} + \boldsymbol{q})\right)}{8m^2 P_4} - \frac{i\gamma \cdot (\boldsymbol{p} + \boldsymbol{q})(\boldsymbol{p} + \boldsymbol{q})^2}{4m^2 P_4^2} + \cdots - \frac{((\boldsymbol{p} + \boldsymbol{q})^2)^3}{8m^3 P_4^4} \right] u(0, s),$$

The left-hand side (lattice diagram)

$$\text{L.H.S } (\mu = 4) = \left[\frac{1}{iP_4} - \frac{\gamma \cdot (\boldsymbol{p} + \boldsymbol{q})}{2m_3 P_4} + \frac{(\boldsymbol{p} + \boldsymbol{q})^2}{2m_2 P_4^2} - \frac{i\gamma \cdot \boldsymbol{q}}{4m_{\alpha_E}^2} + \frac{i\boldsymbol{q}^2 + 2\epsilon_{ijk} \Sigma_i q_j p_k}{8m_E^2 P_4} \right.$$
$$\left. + \frac{i(\boldsymbol{p} + \boldsymbol{q})^2}{8m_{D_\perp^2}^2 P_4} - \frac{i\gamma \cdot (\boldsymbol{p} + \boldsymbol{q})(\boldsymbol{p} + \boldsymbol{q})^2}{4m_2 m_3 P_4^2} + \cdots - \frac{\left((\boldsymbol{p} + \boldsymbol{q})^2\right)^3}{8m_2^3 P_4^4} \right] u(0, s),$$

where $P_4 = \tilde{p}_4 + q_4$.

- The matching is done by adjusting the mass-like terms to be m.
  - Red terms : by the action matching.
  - Blue terms : by the current matching.

Matching

- The matching is done by the condition

$$m_{X,b}(m_0 a, c_j, d_i) = m_b,$$

(index $X = 2, 3, E, \cdots.$) $m_{X,b}(m_0 a, c_j, d_i)$ is symbolized coefficients which depend on parameters c_j in the action, and d_i in the improved current. It gives constraint on the coefficients of the action and current.

- The constraints equations for $\mu = 4$, and for $\mu = i$ should be consistent.

- The results from matching four-quark matrix element should be consistent with those from two-quark matrix element.

- We can interpret the expanded sub-diagram by the HQET description. It can be a good cross-check tool to our calculation.

Cross-check I

- The expanded formula of the right-hand side (continuum QCD) can be described by the HQET Feynman diagram,

$$\Big[R^{(1)}_{\mathrm{HQ},\mu}(p_b+q,p_b) + R^{(0)}_{\mathrm{HQ}}(p_b+q)\sum_{n=0}^{3}\Big(\frac{1}{iP_4}\Lambda^{(0)}_{\mathrm{HQ}}(p_b+q)\Big)^n\frac{1}{iP_4}\Lambda^{(1)}_{\mathrm{HQ},\mu}(p_b+q,p_b)\Big]u(0,s),$$

- $\Lambda^{(0)}_{\mathrm{HQ}}$ and $\Lambda^{(1)}_{\mathrm{HQ}}$ are zero-gluon and one-gluon vertex of HQET Lagrangian.
- $R^{(0)}_{\mathrm{HQ}}$ and $R^{(1)}_{\mathrm{HQ},\mu}$ are zero-gluon and one-gluon vertex from the relation between Dirac field and HQET field.
- Integer n is the number of insersions of the higher order HQET Lagrangians without gluon emission.

Cross-check II

- The expanded formula of the left-hand side (lattice QCD) can be described by the (lattice) HQET Feynman diagram,

$$\Big[R^{\text{lat}(1)}_{\text{HQ},\mu}(p_b+q,p_b) + R^{\text{lat}(0)}_{\text{HQ}}(p_b+q)\sum_{n=0}^{3}\left(\frac{1}{iP_4}\Lambda^{\text{lat}(0)}_{\text{HQ}}(p_b+q)\right)^n \frac{1}{iP_4}\Lambda^{\text{lat}(1)}_{\text{HQ},\mu}(p_b+q,p_b)\Big]u(s,0),$$

- $\Lambda^{\text{lat}(0)}_{\text{HQ}}$ and $\Lambda^{\text{lat}(1)}_{\text{HQ}}$ are zero-gluon and one-gluon vertex of the (lattice) HQET Lagrangian.

- $R^{\text{lat}(0)}_{\text{HQ}}$ and $R^{\text{lat}(1)}_{\text{HQ},\mu}$ lattice counterparts of $R^{(0)}_{\text{HQ}}$ and $R^{(1)}_{\text{HQ},\mu}$.

Cross-check III

- The matching condition requires,

1. The action matching (already done by the OK action matching)

$$\Lambda_{\mathrm{HQ}}^{\mathrm{lat},(0)} \to \Lambda_{\mathrm{HQ}}^{(0)}, \qquad \Lambda_{\mathrm{HQ},\mu}^{\mathrm{lat},(1)} \to \Lambda_{\mathrm{HQ},\mu}^{(1)}.$$

2. The operator matching (determine the improvement parameter)

$$R_{\mathrm{HQ}}^{\mathrm{lat},(0)} \to R_{\mathrm{HQ}}^{(0)}, \qquad R_{\mathrm{HQ},\mu}^{\mathrm{lat},(1)} \to R_{\mathrm{HQ},\mu}^{(1)}.$$

Result I

- The improvement coefficients are as follows,

$$d_1 = \frac{\zeta(1+m_0)}{m_0(2+m_0)} - \frac{1}{2m},$$

$$d_2 = \frac{2\zeta(1+m_0)}{m_0(2+m_0)} d_1 - \frac{r_s\zeta}{2(1+m_0)} - \frac{\zeta^2(1+m_0)^2}{m_0^2(2+m_0)^2} + \frac{1}{4m^2},$$

$$d_E = -\frac{2(1+m_0)\zeta}{m_0^2(2+m_0)^2} - \frac{(m_0+1)\zeta c_E}{m_0(2+m_0)} + \frac{1}{2m^2},$$

$$d_B = \frac{2\zeta(1+m_0)}{m_0(2+m_0)} d_1 - \frac{c_B\zeta}{2(1+m_0)} - \frac{\zeta^2(1+m_0)^2}{m_0^2(2+m_0)^2} + \frac{1}{4m^2},$$

$$d_{r_E} = \frac{1}{16 m_3 m_{\alpha_E}^2} + \frac{d_1 d_E}{4} - \frac{1}{16m^3},$$

Result II

$$d_{EE} = \frac{1+m_0}{(m_0^2+2m_0+2)}\Big[-\frac{1}{4m^3} + \frac{\zeta(1+m_0)(m_0^2+2m_0+2)}{[m_0(2+m_0)]^3} + \frac{\zeta c_E(1+m_0)}{[m_0(2+m_0)]^2} + \frac{(2+2m_0+m_0^2)c_{EE}}{m_0(2+m_0)}\Big],$$

$$d_3 = w_3 - d_1,$$

$$d_4 = \frac{\zeta^3(m_0^3+3m_0^2+5m_0+3)}{2m_0^3(2+m_0)^3} + \frac{r_s\zeta^2(3m_0^2+6m_0+4)}{4m_0^2(2+m_0)^2} + \frac{2(1+m_0)c_2}{m_0(2+m_0)} - \frac{(1+m_0)^2\zeta^2}{2m_0^2(2+m_0)^2}d_1 - \frac{r_s\zeta}{4(1+m_0)}d_1 + \frac{(1+m_0)\zeta d_2}{2m_0(2+m_0)} - \frac{3}{16m^3},$$

$$d_5 = \frac{1}{2}\Big[\frac{\zeta^3(m_0^3+3m_0^2+5m_0+3)}{2m_0^3(2+m_0)^3} + \frac{c_B\zeta^2(3m_0^2+6m_0+4)}{4m_0^2(2+m_0)^2} + \frac{2(1+m_0)c_3}{m_0(2+m_0)} - \frac{(1+m_0)^2\zeta^2}{2m_0^2(2+m_0)^2}d_1 - \frac{c_B\zeta}{4(1+m_0)}d_1 + \frac{(1+m_0)\zeta d_B}{2m_0(2+m_0)} - \frac{3}{16m^3}\Big],$$

Result III

$$d_6 = \frac{2(1+m_0)}{(m_0^2+2m_0+2)}\Big[-\frac{1}{16m_3m_{\alpha E}^2}+\frac{\zeta^2 c_E}{4m_0(2+m_0)}-\frac{\zeta c_{EE}(m_0^2+2m_0+2)}{2m_0(1+m_0)(2+m_0)}$$
$$-\frac{d_E}{4}\left(d_1-\frac{2\zeta(1+m_0)}{m_0(2+m_0)}\right)-\frac{1}{24m_2}+\frac{1}{16m^3}\Big],$$
$$d_7 = \frac{2(1+m_0)}{(m_0^2+2m_0+2)}\Big[-\frac{1}{16m_3m_{\alpha E}^2}+\frac{\zeta^2 c_E}{4m_0(2+m_0)}-\frac{\zeta c_{EE}(m_0^2+2m_0+2)}{2m_0(1+m_0)(2+m_0)}$$
$$-\frac{d_E}{4}\left(d_1-\frac{2\zeta(1+m_0)}{m_0(2+m_0)}\right)-\frac{1}{24m_B}+\frac{1}{16m^3}\Big],$$

Discrepancy in d_E

- The problem on d_E is that our result for d_E is different from that in [El-Khadra, Kronfeld, and Mackenzie (1997)]. They are different from each other even at the leading order in the continuum limit.

$$
\begin{aligned}
d_E(\mathrm{FNAL}) &= \frac{\zeta(1-c_E)(1+m_0 a)}{m_0 a(2+m_0 a)} - \frac{d_1}{m_2 a}. \\
&\to \frac{1}{16}\left(3-2r_s-r_s^2\right) + \frac{1}{48}\left(3-2r_s+3r_s^2\right) m_0 a + \mathcal{O}\left((m_0 a)^2\right), \\
d_E(\mathrm{SWME}) &= -\frac{2(1+m_0 a)\zeta}{m_0^2 a^2 (2+m_0 a)^2} - \frac{\zeta c_E(1+m_0 a)}{m_0 a(2+m_0 a)} + \frac{1}{2 m_2^2 a^2} \\
&\to \frac{1}{48}\left(1-6r_s-3r_s^2\right) + \frac{1}{48}\left(-1+2r_s+3r_s^2\right) m_0 a + \mathcal{O}\left((m_0 a)^2\right)
\end{aligned}
$$

- The Fermilab action is valid for all the range of $m_0 a$. To check, we can study improvement assuming $m_0 a \ll 1$.

Discrepancy in d_E

- In the $m_0 a \ll 1$ limit, one can expand the OK action in the power of a. Let us expand the OK action up to the second order (which is enough to compare d_E)

$$S_{\mathrm{OK},a^2} \doteq \int d^4x \; \bar{\psi}(x) \Big[m_0 + \gamma_4 D_4 + \zeta \boldsymbol{\gamma} \cdot \boldsymbol{D} - \frac{1}{2} a D_4^2 - \frac{1}{2} r_s \zeta a \boldsymbol{D}^2 - \frac{1}{2} c_B \zeta i a \boldsymbol{\Sigma} \cdot \boldsymbol{B} - \frac{1}{2} c_E \zeta a \boldsymbol{\alpha} \cdot \boldsymbol{E} + \frac{1}{6} \gamma_4 a^2 D_4^3 + (c_1 + \frac{1}{6}\zeta) \sum_i \gamma_i a^2 D_i^3 + c_2 a^2 \{\boldsymbol{\gamma} \cdot \boldsymbol{D}, \boldsymbol{D}^2\} + c_3 a^2 \{\boldsymbol{\gamma} \cdot \boldsymbol{D}, i \boldsymbol{\Sigma} \cdot \boldsymbol{B}\} + c_{EE} a^2 \{\gamma_4 D_4, \boldsymbol{\alpha} \cdot \boldsymbol{E}\} \Big] \psi(x). \quad (3)$$

- For the improvement, the action should be equivalent to the Dirac action through $\mathcal{O}(a^2)$,

$$\bar{\psi}(x) \bar{\mathcal{R}} \Big[m + \boldsymbol{\gamma} \cdot \boldsymbol{D} + \gamma_4 D_4 \Big] \mathcal{R} \psi(x) = \text{R.H.S of (3)}$$

where the transformation $\mathcal{R}$ and $\bar{\mathcal{R}}$ are in terms of $m_0 a$, $\boldsymbol{\gamma} \cdot \boldsymbol{D}$, and $\gamma_4 D_4$.

Discrepancy in d_E I

- The matching condition gives constraints to the parameters in the action up to $\mathcal{O}(a^2)$. First of all, it gives condition on ζ as follow

$$\zeta = 1 + \frac{1}{2}(1 - r_s)m_0 a + \frac{1}{24}(-1 + 6r_s + 3r_s^2)m_0^2 a^2 + \mathcal{O}(m_0 a)^3, \qquad (4)$$

  which implies $m_1 = m_2$ in the Fermilab action.
- And it gives constraints to the other parameters. Assuming $m_0 a \ll 1$, it is consistent with the results in [Oktay and Kronfeld (2008)]

$$c_B = r_s, \qquad c_E = \frac{1}{2}(1 + r_s) + \frac{1}{12}\big(-2 - 3r_s + 3r_s^2\big)m_0 a + \mathcal{O}(m_0 a)^2,$$

$$c_1 = -\frac{1}{6} + \mathcal{O}(m_0 a), \qquad c_2 = c_3 = \frac{1}{48}\big(-1 - 6r_s + 3r_s^2\big) + \mathcal{O}(m_0 a),$$

$$c_{EE} = \frac{1}{96}\big(5 + 6r_s - 3r_s^2\big) + \mathcal{O}(m_0 a).$$

Discrepancy in d_E II

- After eliminating $\gamma_4 D_4$ by using the equation of motion, we obtain

$$\mathcal{R} = (1+m_0 a)^{1/2}\Big[1 + \Big(\frac{1}{4}(1-r_s) + \frac{1}{48}\big(1+3r_s^2\big)m_0 a\Big)a\gamma\cdot\boldsymbol{D} + \frac{1}{32}\big(1-10r_s+r_s^2\big)\big(a\gamma\cdot\boldsymbol{D}\big)^2 + \frac{1}{96}\big(1-6r_s-3r_s^2\big)a^2\boldsymbol{\alpha}\cdot\boldsymbol{E}\Big],$$

by identifying $\mathcal{R}$ to the transformation for improved quark field, we obtain the leading behaviors of d_1, d_2, d_B, and d_E as

$$d_1 = \frac{1}{4}(1-r_s) + \frac{1}{48}\big(1+3r_s^2\big)m_0 a + \mathcal{O}\big((m_0 a)^2\big)$$

$$d_2 = d_B = \frac{1}{32}\big(1-10r_s+r_s^2\big) + \mathcal{O}(m_0 a)$$

$$d_E = \frac{1}{48}\big(1-6r_s-3r_s^2\big) + \mathcal{O}(m_0 a).$$

here d_E is consistent with d_E(SWME), but different from d_E(FNAL).

Summary

- We improve current up to $\mathcal{O}(\lambda^3)$ at the tree-level using improved quark field with eleven non-zero coefficients.(Jaehoon Leem, Jon Bailey, Sunkyu Lee)

- We are using the improved current in the calculation of the $\bar{B} \to D^{(*)}\ell\bar{\nu}$ form factors using Oktay-Kronfeld action. [arXiv:1812.07675]

- We expect that the improved quark field will be used for every lattice QCD simulation with the OK heavy quarks.

- The radiative correction is the next step.
  - To reduce radiative correction from overall factor
    $\to$ calculate $\rho_{A_j} = \sqrt{\dfrac{Z_{A_j^{bc}} Z_{A_j^{cb}}}{Z_{V_4^{bb}} Z_{V_4^{cc}}}}$: one-loop (or) non-perturbative method.
  - Action (c_B) and current (d_1, d_2, d_B).